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EFFICIENT NUMERICAL TECHNIQUE FOR DETERMINING TWO-DIMENSIONAL TEMPERATURE DISTRIBUTION FOR *N* ARBITRARILY LOCATED HEAT RECEIVERS WITH PHASE-CHANGE

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Abstract—The new, very fast numerical technique for determining two-dimensional temperature distribution for *n* arbitrarily located heat receivers with phase-change is proposed. The algorithm is based on the simple explicit iteration scheme, and the phase-change problem is solved according to heat balance. Some numerical examples and possibilities of applications are discussed. This method can be used in predicting the shape of a frozen region and its time changes, with the region forming during rock-freezing before shaft-sinking.

NOMENCLATURE

- c_1, specific heat of the medium before the phase-change;
- c_2, specific heat of the medium after the phase-change;
- L, volumetric latent heat of fusion;
- p, ratio of the surface of this part of the grid mesh which changed the state of aggregation to the whole surface of the grid mesh;
- Q_n, heat quantity abstracted out of the grid mesh during the n iteration step;
- Q_p, heat quantity which should be abstracted out of a grid mesh to make the medium inside the mesh change the state of aggregation completely;
- T_p, temperature of the phase-change of the medium;
- T_1, temperature of the medium before the phase-change;
- T_2, temperature of the medium after the phase-change;
- $T^{(n)}_{i,k}$, temperature of the medium inside the (i, k) grid mesh in n iteration step;
- $\partial T/\partial n$, normal derivative of the temperature;
- t, time;
- $\Delta t = t^{(n+1)} - t^{(n)}$, time interval between iteration steps.

Greek letters

- α, thermal diffusivity;
- α_1, thermal diffusivity of the medium before the phase-change;
- α_2, thermal diffusivity of the medium after the phase-change;
- Δ^2, surface of a grid mesh;
- $\Delta'^2_{(n)}$, surface of this part of a grid mesh which changed the state of aggregation during the n iteration step;
- ρ_1, density of the medium before the phase-change;
- ρ_2, density of the medium after the phase-change.

1. INTRODUCTION

In many technical processes it is necessary to at least have approximate information about the temperature distribution in media with arbitrarily located heat receivers with the phase-change. Therefore it is necessary to look for algorithms describing such a process with the accuracy sufficient from the technical point of view.

The algorithm presented in this paper has been constructed to predict the shape of the frozen region and its time changes, with the region which forms during rock freezing before the shaft sinking. In the freezing process freezing holes (heat receivers) are located approximately on the circle diameter 10–20 m. The freezing process, lasting a few months causes the formation of a frozen rock ring which protects against water flooding during mining works.

There are many algorithms offering numerical solutions to the cooling down problem with the phase change [1–5]. In the above-mentioned case, however, where the region considered is large and the time of the process is long, most of them cannot be applied on economical grounds.

The algorithm presented here is characterized by a simplicity which makes the time of its computer execution comparatively short. The results' accuracy makes it possible to put this algorithm into practice for simulating the freezing process.

2. THE ALGORITHM DESCRIPTION

We are considering a homogeneous and isotropic two-dimensional continuous medium. In this medium there are n arbitrarily located heat receivers.

We assume that:

heat receivers have finite sizes;

during the whole process the temperature of the receivers is constant;

the temperature of the receivers is lower than the temperature of the phase-change of the medium;

heat contact of the receivers with environment is ideal;

at the time of starting the process the medium temperature is constant;

far enough from the heat receivers the temperature distribution is time constant.

To solve the problem of the temperature distribution and its time variations for such a model it is enough to solve general conduction equations separately for those regions of the medium where the temperature is lower and higher than the phase-change temperature with the continuity condition on the phase boundary.

Conducting equations are

$$\frac{\partial T_i}{\partial t} = \alpha_i \left(\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} \right) \quad i = 1,2\,; \tag{1a}$$

continuity condition

$$\alpha_1 \frac{\partial T_1}{\partial n} = \alpha_2 \frac{\partial T_2}{\partial n} \bigg|_{\text{on the phase-change boundary}} \tag{1b}$$

To find a numerical solution to these equations we used a finite-difference method. We chose 'a simple explicit iteration scheme', which in the case of a quadratic grid looks like this [6]:

$$T_{i,k}^{(n+1)} = T_{i,k}^{(n)} + \frac{\alpha \Delta t}{\Delta^2} \Big/ [T_{i+1,k}^{(n)} + T_{i-1,k}^{(n)} + T_{i,k+1}^{(n)} + T_{i,k-1}^{(n)} - 4T_{i,k}^{(n)}] \tag{2}$$

with the convergence condition

$$\Delta t \leqslant 0.25 \frac{\Delta^2}{\alpha}. \tag{3}$$

Equation (2) cannot be applied to grid meshes across which goes a phase-change boundary because a constant thermal diffusivity has been assumed in the formula. For such cases we use the following modification

$$T_{i,k}^{(n+1)} = T_{i,k}^{(n)} + \frac{\Delta t}{\Delta^2} [\alpha^{(1)}(T_{i+1,k}^{(n)} - T_{i,k}^{(n)}) + \alpha^{(2)}(T_{i-1,k}^{(n)} - T_{i,k}^{(n)}) + \alpha^{(3)}(T_{i,k+1}^{(n)} - T_{i,k}^{(n)}) + \alpha^{(4)}(T_{i,k-1}^{(n)} - T_{i,k}^{(n)})] \tag{4}$$

in which coefficients $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$, $\alpha^{(4)}$ are selected according to the state of aggregation of the grid meshes which are in contact with the grid mesh (i, k).

The following are possibilities of grid mesh contact:

Both meshes are before the phase-change. Then the sufficient coefficient of thermal diffusivity in formula (4) is the same as the thermal diffusivity coefficient of the medium before the phase-change.

Both meshes are after the phase-change. The thermal diffusivity coefficient for such contact is the same as the thermal diffusivity coefficient of the medium after the phase-change.

Both meshes are in phase-change. The choice of a thermal diffusivity coefficient is of no importance because of the zero temperature difference.

One of the grid meshes is before the phase-change and the other one is in or after the phase-change. We assume that the sufficient thermal diffusivity coefficient in equation (4) is the same as the thermal diffusivity coefficient of the medium before the phase-change.

One of the grid meshes is in the phase-change and the other one is after the phase-change. In this case we assume that the sufficient thermal diffusivity coefficient in (4) is

$$\alpha = \alpha_2 p + \alpha_1 (1 - p). \tag{5}$$

In making an analysis of the phase-change assume that, for the grid mesh (i, k), in the nth iteration step $T_{i,k}^{(n)} > T_p$ and that the temperatures of the adjacent meshes are such that the temperature $T_{i,k}^{(n+1)}$ calculated according to equation (4) is smaller than T_p. That means that the phase change should start in this grid mesh in the interval of time $t^{(n+1)} - t^{(n)}$. Therefore heat quantity Q_{n+1} abstracted out of the grid mesh in $(n+1)$ iteration steps will not lower the temperature below T_p, but will cause a partial or complete change of the state of aggregation in the grid mesh. This heat quantity is

$$Q_{n+1} = |T_{i,k}^{(n+1)} - T_p| \rho_1 \Delta^2 c_1. \tag{6}$$

The heat quantity which should be abstracted out of the grid mesh to make the medium inside the mesh change the state of aggregation completely is

$$Q_p = \rho_1 \Delta^2 L. \tag{7}$$

We can distinguish two cases:

(a) $Q_{n+1} \geqslant Q_p$.

In this case of $(n+1)$ iteration steps the medium inside the mesh changed the state of aggregation and the difference of heat quantity $Q_{n+1} - Q_p$ causes further lowering of the grid mesh temperature according to the formula

$$T_{i,k}^{(n+1)} = T_p - \frac{Q_{n+1} - Q_p}{\rho_2 c_2 \Delta^2}. \tag{8}$$

(b) $Q_{n+1} < Q_p$.

In this case of $(n+1)$ iteration steps a part of the mesh surface determined by

$$\Delta'^2_{(n+1)} = \Delta^2 \frac{Q_{n+1}}{Q_p} \tag{9}$$

changed the state of aggregation. In the next iteration steps we regard this grid mesh as a mesh in the phase-change with the temperature equal to the phase change temperature T_p, calculated [according to the formula (9)] in every iteration step the surface which changes the state of aggregation. If for the $(n+m)$ iteration steps

$$\sum_{i=n+1}^{n+m} \Delta_i'^2 > \Delta^2$$

we assume that the whole surface of the grid mesh changed the state of aggregation then the difference

$$\sum_{i=n+1}^{n+m} Q_i - Q_p$$

causes lowering of the mesh temperature as in case (a).

Such a model of phase-change requires choosing the time step in such a way that between two following iterations a phase-change boundary moves one grid mesh forward at most. This condition is generally fulfilled if the convergence condition (3) of the simple explicit iteration scheme is fulfilled.

The simple explicit iteration scheme has a precise physical interpretation based on the principle of conservation of energy which corresponds with the suggested method of introducing the phase-change. This feature and also the numerical simplicity of the scheme made us decide to use the scheme to solve the given problem.

3. COMPUTATIONAL EXAMPLES AND APPLICATIONS

(1) To check the accuracy of results obtained while using the presented algorithm we have computed the temperature distribution as a function of time for the case given in [4]. The grid mesh was 20 cm wide. All physical properties determining the process were established according to the literature [4].

Figure 1 presents positions of phase-change boundary after 1000, 2000 and 5000 h, respectively. (Continuous line–phase-change boundary positions taken from the literature, stepped line–phase-change boundary positions simulated by means of the method presented above.)

(2) To test the time efficiency of the algorithm we have computed the temperature distribution as a function of time in square 30 × 30 m with the grid mesh being 20 cm wide. Inside the region were located 33 heat receivers with the temperature at 237.5 K; the temperature of the medium was 289 K at the beginning of the process and the temperature of the phase-change of the medium was 271 K. Moreover

$$\alpha_1 = 0.264 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$$
$$\alpha_2 = 0.540 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$$
$$c_1 = 1.398 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$$
$$c_2 = 1.147 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$$
$$\rho_1 = \rho_2 = 2.08 \times 10^3\,\mathrm{kg\,m^{-3}}$$
$$L = 45.22 \times 10^6\,\mathrm{J\,m^{-3}}.$$

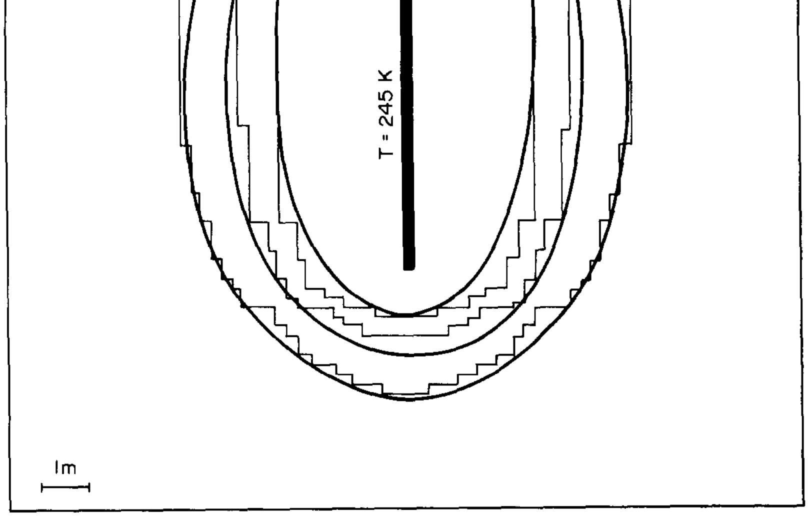

FIG. 1. Phase-change boundary positions after 1000, 2000 and 5000 h, respectively, from the beginning of cooling process. Stepped line—positions simulated by means of the presented method, continuous line—positions given in [4]. Thick black vertical line represents the heat receivers system.

The size of the region as well as the number and location of heat receivers and physical properties determining the process were chosen according to the real conditions observed while freezing the rock before shaft sinking.

Computing has been carried for the period of 5000 h with the time interval (which fulfils the convergence condition) of 4 h. The calculation time of CDC 6000 computer was 2094 s that is 1.7 s per iteration.

(3) For the process of freezing the rock it is important from the technical point of view to have the information concerning the location of phase-change boundary in a given time moment. This problem cannot often be solved by means of simulating temperature distribution because of the lack of sufficiently accurate information on thermal properties of the medium, especially on thermal diffusivities values. As the calculation, with the above-presented algorithm applied, has shown however, the decisive and essential factor for the technical application influence on the isotherm geometry is merely the geometry of the heat receivers' location. Consequently, if inside the freezing region there is one hole allowing measurement of the rock temperature in a given time moment, then by means of simulating the process of phase-change boundary shift it will be possible to determine boundary location in this time moment, even if based on inaccurate values of thermal diffusivities.

(4) The above-presented method of optimum simulation of the process of making a freezing jacket has been applied to the analysis of particular freezing holes' participation in the freezing jacket growth. Taking this method into consideration we can evaluate the efficiency of the heat receivers' configuration and the optimum (for the jacket growth speed) space location of the heat receivers.

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TECHNIQUE NUMERIQUE PERFORMANTE POUR DETERMINER LA DISTRIBUTION BIDIMENSIONNELLE DE TEMPERATURE POUR N RECEPTEURS DE CHALEUR ARBITRAIREMENT SITUES AVEC CHANGEMENT DE PHASE

Résumé—On propose la nouvelle technique numérique, très rapide, pour déterminer la distribution bidimensionnelle de température pour n récepteurs localisés arbitrairement, avec changement de phase. L'algorithme est basé sur un schéma simple à itération explicite et le problème de changement de phase est résolu à partir du bilan de chaleur. On discute des exemples numériques et des possibilités d'application. Cette méthode peut être utilisée pour prédire la forme de la région gelée et sa modification dans le temps.

NEUES NUMERISCHES VERFAHREN FÜR DIE BERECHNUNG DER ZWEIDIMENSIONALEN TEMPERATURVERTEILUNG BEI *N*-WILLKÜRLICH VERTEILTEN WÄRMESENKEN MIT PHASENWECHSEL

Zusammenfassung—Für die Berechnung der zweidimensionalen Temperaturverteilung bei n willkürlich verteilten Wärmesenken mit Phasenwechsel wird ein neues, sehr schnelles numerisches Verfahren angegeben. Der Algorithmus basiert auf einem einfachen expliziten Iterationsverfahren und die Phasenänderung wird in der Wärmebilanz erfaßt. Es werden einige numerische Beispiele und Anwendungsmöglichkeiten diskutiert. Für eine Gefrierzone die sich während des Einfrierens von Gestein vor der Schachtabteufung bildet, kann mit dieser Methode die Form der Zone und ihre zeitliche Änderung vorausgesagt werden.

ЭФФЕКТИВНЫЙ ЧИСЛЕННЫЙ МЕТОД ОПРЕДЕЛЕНИЯ ДВУМЕРНОГО ПОЛЯ ТЕМПЕРАТУР В n ПРОИЗВОЛЬНО РАСПОЛОЖЕННЫХ ПРИЕМНИКАХ ТЕПЛА ПРИ ФАЗОВЫХ ПРЕВРАЩЕНИЯХ

Аннотация — Предложен новый устойчивый численный метод определения двумерных полей температур в n произвольно расположенных тепловых приемниках при фазовых превращениях. Алгоритм базируется на простой явной итерационной схеме, а задача о фазовых превращениях решается методом теплового баланса. Рассматриваются некоторые численные примеры и возможности приложений. Метод может использоваться для расчета формы замерзшей области и изменения ее во времени, а также области, образующейся в замерзающей породе перед проходкой шахтного ствола.